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Solution of Difference Equations by Use of the τ-Method

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In the field of differential equations, the τ-method introduced by C. Lanczos has generated considerable interest because of its novel philosophy. That is, rather than attempting to solve an exact equation approximately, it solves an approximate equation exactly. The τ-method when applied to differential equations has many striking properties. In this paper, the concept is applied to difference equations. For a model we use the equation satisfied by the reciprocal of the gamma function, $1/\Gamma(z+1)$. As a consequence of the analysis, we show how to generate the Taylor series coefficients in the expansion of this function about $z=0$. In particular, a novel technique is provided to compute Euler's constant.

I. INTRODUCTION

The τ-method, due to C. Lanczos, was originally devised to obtain polynomial approximations to the solutions of ordinary linear homogeneous differential equations; see Refs. [1, 2], or the volume by Luke [3], which contains a survey of recent developments and many examples.

The τ-method has generated much interest because of its novel philosophy: rather than attempting to solve an exact equation approximately, it solves an

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approximate equation exactly. Let the original differential equation be

$$L[h(z)]=0, \qquad h^{(j)}(0)=c_j, \qquad 0\leqslant j\leqslant r-1, \tag{1}$$

where r is the order of L and where the origin is a regular point of the equation. In its most naive formulation the τ-method takes as an approximate equation

$$L[h_n(z)]=\tau p_n(z/\sigma), \tag{2}$$

where p_n is a given polynomial of degree n, $\sigma\neq 0$ a given range parameter and τ a constant to be determined. If the τ-method in this formulation works, $h_n(z)$ will be a polynomial of degree n, a solution of the approximate equation which also satisfies the initial conditions for the initial problem. Depending on the problem, it may be necessary to add other τ-terms, e.g.,

$$\sum_{j=0}^{r} \tau_j p_{n+j}(z/\sigma) \tag{3}$$

the number of these being dictated by the requirement that the linear equations for the determination of the coefficients of $h_n(z)$ be consistent.

One very useful property of the method is that the error $h_n(z)-h(z)$, satisfies the non-homogeneous equation (2). In many important cases the error can be analyzed by applying the method of variation of parameters to this equation.

Generally, the polynomials p_n are chosen to be interpolatory in the range $[0, 1]$ and σ is chosen so that $[0, \sigma]$ is the largest interval over which the approximation is required. Often p_n is taken to be the Jacobi polynomial $P_n^{(\alpha,\beta)}$ shifted to $[0, 1]$.

All in all, the application of the τ-method to differential equations is rather straightforward. The existence and uniqueness of polynomial solutions is easily assured, even polynomial solutions having prescribed initial data. Often convergence as $n\to\infty$ can be shown.

Instead of solving (2), we could seek a polynomial solution of this same equation with $\tau p_n(z/\sigma)$ replaced by a single power of z, say z^n. This idea goes back to Lanczos [1], and was subsequently studied by Ortiz [4]. The solution is called a canonical polynomial. Then the solution to (2) with appropriate linear initial conditions is a particular combination of the canonical polynomials. The scheme can be advantageous since the canonical polynomials used to obtain the solution of (2) are available to obtain the solution of (2) with n replaced by $n+1$.

Since several important higher transcendental functions satisfy difference rather than differential equations, we sought to extend the application of the τ-method to these kinds of equations. We encountered problems immediately.

For instance, there is no theory guaranteeing polynomial solutions of the related approximate equation. In fact, in the example treated in this paper, the functions satisfying the approximate equation are meromorphic, with simple poles, some in the range where the approximation is to hold. Fortunately (and we are not certain whether this is simply a characteristic of our example) it is possible to choose the τ_j so the residues of the solution are zero, and further, to restrict the growth of the solution. The effect of this construction is to yield a polynomial solution.

The body of our paper consists of a single example, the equation for the reciprocal of the gamma function $1/\Gamma(z+1)$ and numerics. Many of the peculiarities of this example, we feel, will apply to other difference equations and point the way for future investigations.

II. The τ-Method

Consider the difference equation

$$(z+1)\,h(z+1) - h(z) = 0 \tag{4}$$

which is satisfied by

$$h(z) = \frac{C(z)}{\Gamma(z+1)}, \tag{5}$$

where $\Gamma(z)$ is the gamma function and $C(z)$ is an arbitrary periodic function with period 1. We take $C(z) = 1$. The reciprocal of the gamma function is entire so we have the series representation

$$h(z) = \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} h_k z^k, \qquad h_0 = 1, |z| < \infty. \tag{6}$$

The coefficients h_k, $k = 1(1)\,29$ to $20d$ are available in the volumes by Luke [3, 5].

In the spirit of the τ-method, we shall study the difference equation

$$\begin{aligned}(z+1)\,h_n(z+1) - h_n(z) &= \tau_1 A_n R_n^{(\alpha,\beta)}(z/\sigma) \\ &\quad + \tau_2 A_{n+1} R_{n+1}^{(\alpha,\beta)}(z/\sigma), \qquad 0 < z/\sigma \leqslant 1, \\ A_n R_n^{(\alpha,\beta)}(x) &= A_n P_n^{(\alpha,\beta)}(2x-1) \\ &= \sum_{k=0}^{n} \frac{(-)^k \binom{n}{k} (n+\lambda)_k x^k}{(\beta+1)_k},\end{aligned} \tag{7}$$

$$A_n = \frac{(-)^n n!}{(\beta+1)_n}, \qquad \lambda = \alpha + \beta + 1. \tag{8}$$

We look for a solution of (7) in the form

$$h_n(z) = \sum_{k=0}^{n} h_{n,k} z^k, \qquad h_{n,0} = 1. \tag{9}$$

Note from (4) and (7) that $\varepsilon_n(z) = h_n(z) - h(z)$ also satisfies (7).

Because of the additive property of particular solutions of (7), if we can determine a solution of

$$(z+1)\, g(z+1) - g(z) = z^k, \qquad k = 0, 1, 2, \ldots, \tag{10}$$

then a particular solution of (4) readily follows. This approach is analogous to the canonical polynomial scheme of Lanczos for differential equations. Let

$$g(z) = \varphi(z)/\Gamma(z+1). \tag{11}$$

Then

$$\varphi(z+1) - \varphi(z) = z^k \Gamma(z+1). \tag{12}$$

One might hope to solve this equation by applying the Nörlund sum operator, but the Nörlund sum of the function on the right-hand side does not converge. To get around this, replace z by $-z$ and take $\theta(z+1) = \varphi(-z)$. Then

$$\theta(z+1) - \theta(z) = -\Gamma(1-z)(-z)^k. \tag{13}$$

The Nörlund sum of the function on the right now converges. We have for a particular solution

$$\theta(z) = \sum_{j=0}^{\infty} \Gamma(1-z-j)(-z-j)^k, \qquad z \neq 0, \pm 1, \pm 2, \ldots. \tag{14}$$

Thus a particular solution of (10) is

$$P_k(z) = \sum_{j=0}^{\infty} \frac{(-)^{j+1}(z-1-j)^k}{(-z)_{j+1}}, \qquad z \neq 0, 1, 2, \ldots. \tag{15}$$

It is easy to see from (15) that $P_0(z)$ is a meromorphic function whose only singularities are simple poles at $z = 0, 1, 2, \ldots$. Also $P_1(z) = 1$. If $k > 1$, $P_k(z)$ has the same singular behavior as $P_0(z)$ and except for the singular part, $P_k(z)$ as we shall show, is a polynomial in z of degree $(k-1)$. Thus we can write

$$P_k(z) = \sum_{m=0}^{\infty} \frac{d_{k,m}}{z-m} + Q_k(z),$$

$$Q_k(z) = \sum_{m=0}^{k-1} q_{k,m} z^m, \qquad q_{k,0} \equiv q_k, \tag{16}$$

$$Q_0(z) = 0, d_{1,m} = 0 \qquad \text{for all } m.$$

Then

$$\begin{aligned} d_{k,m} &= \text{Residue } P_k(z) = \lim_{z\to m} (z-m) P_k(z) \\ &= \lim_{z\to m} \sum_{j=0}^{\infty} \frac{(-)^j (z-1-j)^k}{(-z)(1-z)\cdots(m+1+z)(m+1-z)(m+2-z)\cdots(j-z)} \\ &= S_k/m! \end{aligned} \tag{17}$$

$$S_k = (-)^k \sum_{j=0}^{\infty} \frac{(-)^j (j+1)^k}{j!}, \qquad S_0 = e^{-1}, S_1 = 0.$$

Define

$$u_k = \lim_{z\to 0} \left\{ \frac{d}{dz} [zP_k(z)] \right\} = q_k - S_k \sum_{m=1}^{\infty} \frac{1}{m(m!)}, \qquad q_k \equiv q_{k,0}. \tag{18}$$

Taking the limit directly in (15) gives

$$u_k = (-)^k \sum_{j=0}^{\infty} \frac{(-)^j (j+1)^k}{j!} \left[-\frac{k}{j+1} + \psi(j+1) - \psi(1) \right], \tag{19}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function.

Comparing (18) and (19) and using some results in Luke [3, Vol. 1, p. 222, Eqs. (15) and (18)], we find that $q_0 = 0$ and to $15d$,

$$u_0 = -e^{-1}[Ei(1) - \gamma] = -0.4848829106995688. \tag{20}$$

Here γ is the Euler–Mascheroni constant. The numerical result in (20) is readily deduced from known tables. Since $q_0 = 0$, from (18),

$$u_k = q_k + S_k e u_0. \tag{21}$$

It will be necessary to compute S_k and q_k for $1 \leqslant k \leqslant n+1$. Obviously use of either (17) or (19) is not a satisfactory way of doing this especially for k large because of round off error.

We now develop some properties of $P_k(z)$ which will be useful and will also lead to more reasonable representations for $Q_k(z)$, S_k and q_k including recurrence formulas. The results can be summarized by the following theorem.

THEOREM 1. *The functions $P_k(z)$ satisfy the recurrence formula*

$$P_{k+1}(z) = (z-1)^k + (-)^k \sum_{r=0}^{k-1} (-)^r \binom{k}{r} P_r(z),$$
$$P_1(z) = 1, k = 1, 2, \ldots. \quad (22)$$

Further

$$S_{k+1} = (-)^k \sum_{r=0}^{k-1} (-)^r \binom{k}{r} S_r, \qquad k \geqslant 1, \quad (23)$$

$$q_{k+1} = (-)^k + (-)^k \sum_{r=1}^{k-1} (-)^r \binom{k}{r} q_r,$$
$$q_0 = 0, q_1 = 1, k = 2, 3, \ldots. \quad (24)$$

Note. The formulae (22)–(24) are valid for $k = 0$, 0 and 1, respectively, provided the empty sum $\sum_{r=1}^{-1}$ is treated as zero. Also if we let $S_k = e^{-1} s_k$, then the s_k's are integers, and for example,

$$\begin{array}{lll} s_0 = 1, & s_1 = 0, & s_2 = -1, \\ s_3 = 1, & s_4 = 2, & s_5 = -s_6 = -9. \end{array} \quad (25)$$

Proof. Replace k by r in (15), multiply by $t^r \binom{k}{r}$ and sum from $r = 0$ to $r = k$. Then

$$\sum_{r=0}^{k} t^r \binom{k}{r} P_r(z) = \sum_{r=0}^{k} t^r \binom{k}{r} \sum_{j=0}^{\infty} \frac{(-)^{j+1}(z+1-j)^r}{(-z)_{j+1}}.$$

But

$$\sum_{r=0}^{k} t^r \binom{k}{r} (z-1-j)^r = [1 + t(z-1-j)]^k,$$

and so

$$\sum_{r=0}^{k} t^r \binom{k}{r} P_r(z) = t^k \sum_{j=0}^{\infty} \frac{(-)^{j+1}(z-1-j+1/t)^k}{(-z)_{j+1}}. \quad (26)$$

Also from (15),

$$P_k(z)+P_{k+1}(z)=\sum_{j=0}^{\infty}\frac{(-)^{j+1}}{(-z)_{j+1}}\{(z-1-j)^k+(z-1-j)^{k+1}\}$$
$$=\sum_{j=0}^{\infty}\frac{(-)^j(z-1-j)^k}{(-z)_j}$$
$$=(z-1)^k+\sum_{j=0}^{\infty}\frac{(-)^{j+1}(z-2-j)^k}{(-z)_{j+1}}. \tag{27}$$

Combining this with (25) for $t=-1$ gives the statement (22). Equation (23) follows from (17) and (24) emerges when the limit process in (18) is applied to (22) and account is taken of (21) and (23). ∎

We now have the machinery to establish (16) and some related results.

THEOREM 2. (a) *$P_0(z)$ is a meromorphic function whose only singularities are simple poles at $z=0$ and the positive integers*

(b) $P_1(z)=1$.

(c) *$P_k(z)$ has the same singular behavior as $P_0(z)$, $k>1$.*

(d) *Except for the singular part, $P_k(z)$ is a polynomial in z of degree $k-1$, call it $Q_k(z)$ (see (16)),*

$$P_k(z)=S_kP_0(z)/S_0+Q_k(z). \tag{28}$$

(e) *$Q_k(z)$ satisfies the recursion formula*

$$Q_{k+1}(z)=(z-1)^k+(-)^k\sum_{r=0}^{k-1}(-)^r\binom{k}{r}Q_r(z),$$
$$Q_0(z)=0,\ Q_1(z)=1,\ k=1,2,\dots . \tag{29}$$

Proof. Parts (a) and (b) are easily verified from (15). Items (c) and (d) follow by induction using (22). Equations (28) and (29) also follow by induction using (22) and (23). ∎

It is interesting that the recursion formula naturally gives $Q_k(z)$ as a polynomial in $(z-1)$ of degree $k-1$. We have

$$Q_2(z)=(z-1),\qquad Q_3(z)=(z-1)^2-2=z^2-2z-1,$$
$$Q_4(z)=(z-1)^3-3(z-1)+3=z^3-3z^2+5,$$
$$\begin{aligned}Q_5(z)&=(z-1)^4-4(z-1)^2+6(z-1)+4\\&=z^4-4z^3+2z^2+10z-5,\end{aligned} \tag{30}$$

$$\begin{aligned} Q_6(z) &= (z-1)^5 - 5(z-1)^3 + 10(z-1)^2 + 5(z-1) - 30 \\ &= z^5 - 5z^4 + 5z^3 + 15z^2 - 25z - 21. \end{aligned}$$

The "canonical" polynomials $Q_k(z)$ also satisfy the z difference equation

$$(z+1)\, Q_k(z+1) - Q_k(z) = z^k - S_k/S_0. \tag{31}$$

The following theorem gives an estimate for $P_k(z)$.

THEOREM 3. *Let $N_\rho(a)$ denote the open ball in C with center a, radius ρ and for a fixed δ, $0 < \delta < \frac{1}{2}$, let $D_\delta = C - \bigcup_{k=0}^{\infty} N_\delta(k)$. Then*

$$P_k(z)/z^{k-1} = 1 + O(z^{-1}), \qquad z \to \infty \text{ in } D_\delta. \tag{32}$$

Proof. By virtue of (22), we need only show this for $P_0(z)$. $P_0(z)$ may be expressed in terms of the confluent hypergeometric function.

$$\begin{aligned} zP_0(z) &= 1 + (z-1)^{-1} K(z), \\ K(z) &= {}_1F_1(1; 2-z; -1) = e^{-1}{}_1F_1(1-z; 2-z; 1) \\ &= e^{-1} \sum_{j=0}^{\infty} \left(\frac{1-z}{1-z+j} \right) \frac{1}{j!}, \end{aligned} \tag{33}$$

the latter following from Kummer's transformation formula. Now $(1-z)/(1-z+j) = 1 - j/(1-z+j)$ and so

$$K(z) = e^{-1} \left\{ e - \sum_{j=0}^{\infty} [(j+2-z)\, j!]^{-1} \right\}.$$

Since $|j+2-z| \geqslant \delta$ in D_δ, we have

$$|K(z)| \leqslant (1 + \delta^{-1}) \qquad \text{in } D_\delta, \tag{34}$$

and using this in the first line of (33) gives the required result. ∎

We now return to the original difference equation (7). Using (10) and (15), we have

$$\begin{aligned} h_n(z) &= \tau_1 \sum_{k=0}^{n} w_{n,k} P_k(z) + \tau_2 \sum_{k=0}^{n+1} w_{n+1,k} P_k(z), \\ w_{n,k} &= \frac{(-)^k}{\sigma^k} \binom{n}{k} \frac{(n+\lambda)_k}{(\beta+1)_k}, \qquad \lambda = \alpha + \beta + 1, \end{aligned} \tag{35}$$

which is meromorphic with simple poles at $z = 0, 1, 2, \ldots$ and is $O(z^n)$ in D_δ. We want to impose two conditions on $h_n(z)$: first, that it is a polynomial of degree not greater than n and second, $h_n(0) = 1$. Put (28) in (35) and note (16). Then these conditions translate to give equations

$$\tau_1 \sum_{k=0}^{n} w_{n,k} S_k + \tau_2 \sum_{k=0}^{n+1} w_{n+1,k} S_k = 0,$$
$$\tau_1 \sum_{k=0}^{n} w_{n,k} q_k + \tau_2 \sum_{k=0}^{n+1} w_{n+1,k} q_k = 1. \tag{36}$$

We can then write

$$h_n(z) = \tau_1 \sum_{k=0}^{n} w_{n,k} Q_k(z) + \tau_2 \sum_{k=0}^{n+1} w_{n+1,k} Q_k(z). \tag{37}$$

We summarize the facts concerning the derivation of (36) and (37) as an existence and uniqueness

THEOREM 4. *Let $\alpha > -1$, $\beta > -1$, $n \geqslant 0$ and σ be given. Let the determinant of the system* (36) *be non-zero. Then there exist unique constants τ_1 and τ_2, and a unique polynomial $h_n(z)$ of degree not exceeding n with $h_n(0) = 1$ satisfying* (7). *If $\tau_2 \neq 0$, the degree of $h_n(z)$ is exactly n.*

The coefficients $h_{n,k}$ in $h_n(z)$ (see (9)) can be obtained from (37) with the aid of (29) and (30). This procedure can be avoided by use of a recursion formula for $h_{n,k}$. Put (9) in (7) and equate like powers of z. Then

$$\sum_{r=k-1}^{n} \binom{r+1}{k} h_{n,r} - h_{n,k} = \tau_1 w_{n,k} + \tau_2 w_{n+1,k},$$
$$k = n, n-1, \ldots, 1, \; h_{n,n} = \tau_2 w_{n+1,n+1}. \tag{38}$$

The latter is fascinating for it suggests another way of computing $h_{n,k}$ which avoids use of the S_k's and the q_k's. The procedure which is much akin to the J. C. P. Miller algorithm for solutions of linear recurrence equations is as follows. Compute $f_{n,k}$ and $g_{n,k}$ by recursion using

$$f_{n,k-1} = w_{n,k} - k f_{n,k} - \sum_{r=k+1}^{n} \binom{r+1}{k} f_{n,r},$$
$$g_{n,k-1} = w_{n+1,k} - k g_{n,k} - \sum_{r=k+1}^{n} \binom{r+1}{k} g_{n,r}, \qquad k = n-1, n-2, \ldots, 1,$$
$$f_{n,n} = 0, \qquad f_{n,n-1} = w_{n,n},$$
$$g_{n,n} = w_{n+1,n+1}, \; g_{n,n-1} = w_{n+1,n} - n w_{n+1,n+1}. \tag{39}$$

Then

$$h_{n,k} = \tau_1 f_{n,k} + \tau_2 g_{n,k} \tag{40}$$

and the coefficients are completely determined once we know τ_1 and τ_2. Since $h_n(0) = 1$,

$$\tau_1 f_{n,0} + \tau_2 g_{n,0} = 1. \tag{41}$$

To obtain a second relation involving τ_1 and τ_2, combine (7) and (9) with $z = 0$ and use (40). Then

$$\tau_1 \left(1 - \sum_{k=1}^{n} f_{n,k}\right) + \tau_2 \left(1 - \sum_{k=1}^{n} g_{n,k}\right) = 0. \tag{42}$$

III. Numerics for the τ-Method

In this section we present some numerics to illustrate computation of the approximations for $[\Gamma(z+1)]^{-1}$ based on the scheme (39)–(42). In all the exhibited calculations, $\alpha = \beta = -\frac{1}{2}$ or $\alpha = \beta = 0$, and $\sigma = 1$. The following table relates the machine notation with that of this paper.

Notation

$q_{n,k}$, where	Q(N, K) where
$q = w, f, g$ or h	Q = W, F, G, or H
τ_1, τ_2	TAU1, TAU2
z	Z
$h_n(z)$	HN(Z)

Computations were done for $n = 2(1)\,15$. The printouts for $n = 5$ and 10 with $\alpha = \beta = -\frac{1}{2}$ are given in Table I.

Coefficients in the Taylor series expansion of $1/\Gamma(z+1)$ about $z = 0$ will be found in either of the volumes by Luke [3, 5]. Clearly the series converges for all z. Let $0 \leqslant z \leqslant 1$. If $1/\Gamma(z+1) = \sum_{k=0}^{\infty} h_k z^k$ is approximated by the sum of the first $(n+1)$ terms, then the magnitude of the error is bounded by $E_n = \sum_{k=n+1}^{\infty} |h_k|$. Values for E_n for $n = 2(1)\,15$ are given in Table II. For the same n, we also give the magnitude of the maximum errors obtained by the τ-method for $0 \leqslant z \leqslant 1$ based on the errors for $z = 0$, 1/4, 1/3, 1/2, 2/3, 3/4, 1 (see Table I, for example). These data are notated as F_n and are presented in Table II for $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = 0$. We have not been able

SOLUTION OF A DIFFERENCE EQUATION

N= 5 SIGMA=1. ALPH=-0.50 BETA=-0.50

K	W(N,K)	W(N+1,K)	F(N,K)	G(N,K)	H(N,K)
0	-----	-----	0.963000000000000D 04	0.249440000000000D 05	0.100000000000000D 01
1	-0.500000000000000D 02	-0.720000000000000D 02	-0.248000000000000D 04	-0.104632000000000D 06	0.583524058009239D 00
2	0.400000000000000D 03	0.840000000000000D 03	-0.598400000000000D 04	-0.588800000000000D 04	-0.703708318119733D 00
3	-0.112000000000000D 04	-0.358400000000000D 04	0.332800000000000D 04	0.417280000000000D 05	0.620533272782439D-01
4	0.128000000000000D 04	0.691200000000000D 04	-0.512000000000000D 03	-0.163840000000000D 05	0.757874141237157D-01
5	-0.512000000000000D 03	-0.614400000000000D 04	0.0	0.204800000000000D 04	-0.175390203662418D-01
6	-----	0.204800000000000D 04	-----	-----	-----

TAU1= 0.126024900012145D-03 TAU2=-0.856397478820399D-05

Z	HN(Z)	1/GAMMA(1+Z)	ERROR
0.0	0.100000000000000D 01	0.100000000000000D 01	0.0
0.250000000000000D 00	0.110314774449539D 01	0.110326265132084D 01	0.114906825444239D-03
0.333333333333333D 00	0.111947994764208D 01	0.111984652172219D 01	0.366574080109094D-03
0.500000000000000D 00	0.112778023438075D 01	0.112837916709551D 01	0.598932714759304D-03
0.666666666666667D 00	0.110730364539431D 01	0.110773216743247D 01	0.428522038161239D-03
0.750000000000000D 00	0.108780338176310D 01	0.108806525213102D 01	0.261870367914030D-03
0.100000000000000D 01	0.100011746092522D 01	0.100000000000000D 01	-0.117460925223689D-03

SOLUTION OF A DIFFERENCE EQUATION

N=10 SIGMA=1. ALPH=-0.50 BETA=-0.50

K	W(N,K)	W(N+1,K)	F(N,K)	G(N,K)	H(N,K)
0	-----	-----	0.336891198400000D 10	0.103864250398000D 12	0.100000000000000D 01
1	-0.200000000000000D 03	-0.242000000000000D 03	0.436977812000000D 10	0.285289819200000D 10	0.577213336569068D 00
2	0.660000000000000D 04	0.968000000000000D 04	-0.244099200000000D 10	-0.626748884160000D 11	-0.655881412595311D 00
3	-0.844800000000000D 05	-0.151008000000000D 06	-0.934151936000000D 09	0.143046919680000D 11	-0.419747251923968D-01
4	0.549120000000000D 06	0.120806400000000D 07	0.842678272000000D 09	0.106551198720000D 11	0.166474663528542D 00
5	-0.205004800000000D 07	-0.563763200000000D 07	-0.877465600000000D 08	-0.561687756800000D 10	-0.419407157830588D-01
6	0.465920000000000D 07	0.164003840000000D 08	-0.901120000000000D 08	0.228425728000000D 09	-0.103349353675865D-01
7	-0.655360000000000D 07	-0.306380800000000D 08	0.406978560000000D 08	0.556859392000000D 09	0.827074110978213D-02
8	0.557056000000000D 07	0.367656960000000D 08	-0.734003200000000D 07	-0.204603392000000D 09	-0.206034051943320D-02
9	-0.262144000000000D 07	-0.273940480000000D 08	0.524288000000000D 06	0.325058560000000D 08	0.244836843061222D-03
10	0.524288000000000D 06	0.115343360000000D 08	0.0	-0.209715200000000D 07	-0.114484586785865D-04
11	-----	-0.209715200000000D 07	-----	-----	-----

TAU1= 0.128528086744500D-09 TAU2= 0.545905050210404D-11

Z	HN(Z)	1/GAMMA(1+Z)	ERROR
0.0	0.100000000000000D 01	0.100000000000000D 01	0.0
0.250000000000000D 00	0.110326217581123D 01	0.110326265132084D 01	0.475509602804181D-06
0.333333333333333D 00	0.111984606390752D 01	0.111984652172219D 01	0.457814669063694D-06
0.500000000000000D 00	0.112837904420044D 01	0.112837916709551D 01	0.122895072340867D-06
0.666666666666667D 00	0.110773243900270D 01	0.110773216743247D 01	-0.271570228971285D-06
0.750000000000000D 00	0.108806560251885D 01	0.108806525213102D 01	-0.350387836522970D-06
0.100000000000000D 01	0.100000000013399D 01	0.100000000000000D 01	-0.133986377548467D-09

TABLE II

Comparison of Errors in the Taylor Series with the τ-Method

n	Taylor Series E_n	τ-Method $F_n, \alpha=\beta=-\frac{1}{2}$	τ-Method $F_n, \alpha=\beta=0$	τ-Method E_n^*
2	0.259	0.109	0.123	0.951(−2)
3	0.218	0.489(−2)	0.513(−2)	0.344(−2)
4	0.605(−1)	0.281(−2)	0.369(−2)	0.997(−3)
5	0.183(−1)	0.599(−3)	0.401(−3)	0.227(−3)
6	0.875(−2)	0.562(−4)	0.513(−4)	0.723(−4)
7	0.153(−2)	0.405(−4)	0.397(−4)	0.504(−4)
8	0.366(−3)	0.117(−4)	0.115(−4)	0.580(−5)
9	0.151(−3)	0.182(−5)	0.178(−5)	0.264(−5)
10	0.227(−4)	0.350(−6)	0.465(−6)	0.889(−6)
11	0.271(−5)	0.433(−6)	0.426(−6)	0.357(−6)
12	0.146(−5)	0.106(−6)	0.105(−6)	0.555(−7)
13	0.218(−6)	0.145(−7)	0.170(−7)	0.635(−7)
14	0.124(−7)	0.148(−7)	0.146(−7)	0.146(−7)
15	0.630(−8)	0.447(−8)	0.443(−8)	0.144(−8)

to prove that for fixed z and $0 < z/\sigma \leqslant 1$, the sequence $h_n(z) \to [\Gamma(z+1)]^{-1}$ as $n \to \infty$. However, the computations offer heuristic evidence that this is so. Also for the n values recorded, $F_n < E_n$ and so overall in the range $0 \leqslant z \leqslant 1$, the τ-method gives a better approximation. Notice that the differences in F_n for $\alpha=\beta=-\frac{1}{2}$ and $\alpha=\beta=0$ are very slight.

As will be shown in the next section, if $\sigma \to 0$, we get an equation like (7), where the right-hand side is replaced by $u_1 z^n + u_2 z^{n+1}$. Further with $n \to \infty$, this leads to a scheme to obtain the Taylor series coefficients in the expansion of $1/\Gamma(z+1)$ about $z=0$. Suppose that E_n^* is the magnitude of the maximum error in this polynomial approximation to $1/\Gamma(z+1)$ for $0 \leqslant z \leqslant 1$ based on the errors for the same z values used above. These data are also recorded in Table II. For all the 14 n-values recorded, E_n^* is less than F_n $(\alpha=\beta=-\frac{1}{2})$ ten times. Further $E_n^* < E_n$ for all n except $n=14$.

IV. A Miller Algorithm for the Computation of the Derivatives of $[\Gamma(z+1)]^{-1}$ at $z=0$

When $\sigma \to 0$, the algorithm described by (36) and (37) yields a method of computing the reduced derivatives

$$h_k = \frac{1}{k!}\frac{d^k}{dz^k}[\Gamma(z+1)]_{z=0}^{-1} \tag{43}$$

by taking a limit as $n \to \infty$. From (36)

$$\tau_1 = -\frac{S_{n+1}}{w_{n,n} v_n} + \cdots = O(\sigma^n), \qquad \sigma \to 0,$$
$$\tau_2 = \frac{S_n}{w_{n+1,n+1} v_n} + \cdots = O(\sigma^{n+1}), \qquad \sigma \to 0, \tag{44}$$
$$v_n = q_{n+1} S_n - q_n S_{n+1}.$$

So as $\sigma \to 0$, (7) becomes

$$(z+1)\, h_n(z+1) - h_n(z) = \tau_1 z^n + \tau_2 z^{n+1}. \tag{45}$$

Hence from our previous work,

$$h_n(z) = \tau_1 Q_n(z) + \tau_2 Q_{n+1}(z),$$
$$\tau_1 = -S_{n+1}/v_n, \qquad \tau_2 = S_n/v_n, \tag{47}$$

where v_n is given in (44). We conjecture that

$$h_k = \lim_{n\to\infty} h_{n,k}, \qquad h_{n,k} = (\tau_1 q_{n,k} + \tau_2 q_{n+1,k}), \tag{48}$$

where h_k and $q_{n,k}$ are defined in (6) and (16), respectively.

Recall that in place of the developments surrounding (35) and (37), we could determine the desired approximation by use of the schema given by Eqs. (39)–(42). The same idea can be used to evaluate $h_{n,k}$. We have

$$f_{n,k-1} = -kf_{n,k} - \sum_{r=k+1}^{n} \binom{r+1}{k} f_{n,r},$$
$$g_{n,k-1} = -kg_{n,k} - \sum_{r=k+1}^{n} \binom{r+1}{k} g_{n,r}, \qquad k = n-1, n-2, \ldots, 1,$$
$$f_{n,n} = 0, \qquad f_{n,n-1} = 1,$$
$$g_{n,n} = 1, \qquad g_{n,n-1} = -n. \tag{49}$$

Then for μ_1 and μ_2 satisfying,

$$\mu_1 f_{n,0} + \mu_2 g_{n,0} = 1,$$
$$\mu_1 \sum_{k=1}^{n} f_{n,k} + \mu_2 \sum_{k=1}^{n} g_{n,k} = 0, \tag{50}$$

we have

$$h_k \sim h_{n,k} = \mu_1 f_{n,k} + \mu_2 g_{n,k}, \qquad n \to \infty. \tag{51}$$

TABLE III

Approximations for the Coefficients in the Taylor Series for $1/\Gamma(z+1)$

SOLUTION OF A DIFFERENCE EQUATION

N=10

K	F(N,K)	G(N,K)	H(N,K)
0	0.551300000000000D 04	-0.132090000000000D 05	0.100000000000000D 01
1	0.600000000000000D 02	0.177700000000000D 05	0.577220490530812D 00
2	-0.331800000000000D 04	0.624100000000000D 04	-0.655874509463605D 00
3	0.789000000000000D 03	-0.774700000000000D 04	-0.420373417402381D-01
4	0.555000000000000D 03	0.753000000000000D 03	0.166517021593514D 00
5	-0.300000000000000D 03	0.110800000000000D 04	-0.421121582342600D-01
6	0.150000000000000D 02	-0.425000000000000D 03	-0.957916512478801D-02
7	0.270000000000000D 02	0.500000000000000D 01	0.710077993475868D-02
8	-0.900000000000000D 01	0.350000000000000D 02	-0.120772297066351D-02
9	0.100000000000000D 01	-0.100000000000000D 02	-0.590091711911745D-04
10	0.0	0.100000000000000D 01	0.316146456615287D-04

MU1= 0.257137285424113D-03 MU2= 0.316146456615287D-04

Z	HN(Z)	1/GAMMA(1+Z)	ERROR
0.0	0.100000000000000D 01	0.100000000000000D 01	0.346944695195361D-15
0.250000000000000D 00	0.110326354039812D 01	0.110326265132084D 01	-0.889077282018036D-06
0.333333333333333D 00	0.111984733086412D 01	0.111984652172219D 01	-0.809141936564117D-06
0.500000000000000D 00	0.112837925742845D 01	0.112837916709551D 01	-0.903329366863659D-07
0.666666666666667D 00	0.110773149952169D 01	0.110773216743247D 01	0.667910780682845D-06
0.750000000000000D 00	0.108806446168615D 01	0.108806525213102D 01	0.790444865339524D-06
0.100000000000000D 01	0.100000000000000D 01	0.100000000000000D 01	0.513478148889135D-15

SOLUTION OF A DIFFERENCE EQUATION

N=15

K	F(N,K)	G(N,K)	H(N,K)
0	0.415110000000000D 05	0.705612850000000D 08	0.100000000000000D 01
1	-0.142102580000000D 08	0.531015000000000D 08	0.577215657084541D 00
2	0.133069100000000D 07	-0.474599050000000D 08	-0.655878074400793D 00
3	0.465165100000000D 07	-0.700849900000000D 07	-0.420025775227498D-01
4	-0.164840600000000D 07	0.131899880000000D 08	0.166538634932832D 00
5	-0.310459000000000D 06	-0.270920700000000D 07	-0.421978743975638D-01
6	0.309405000000000D 06	-0.948225000000000D 06	-0.962202091004130D-02
7	-0.558480000000000D 05	0.558194000000000D 06	0.721911868204537D-02
8	-0.114030000000000D 05	-0.723430000000000D 05	-0.116512162674102D-02
9	0.711900000000000D 04	-0.213930000000000D 05	-0.215378372983392D-03
10	-0.112000000000000D 04	0.100120000000000D 05	0.128028755455139D-03
11	-0.105000000000000D 03	-0.132500000000000D 04	-0.200611772788724D-04
12	0.770000000000000D 02	-0.155000000000000D 03	-0.124751667024987D-05
13	-0.140000000000000D 02	0.900000000000000D 02	0.110246551660549D-05
14	0.100000000000000D 01	-0.150000000000000D 02	-0.200160402557879D-06
15	0.0	0.100000000000000D 01	0.141648343267068D-07

MU1= 0.123121123427231D-07 MU2= 0.141648343267068D-07

Z	HN(Z)	1/GAMMA(1+Z)	ERROR
0.0	0.100000000000000D 01	0.100000000000000D 01	0.277555756156289D-16
0.250000000000000D 00	0.110326265003943D 01	0.110326265132084D 01	0.128141031119355D-08
0.333333333333333D 00	0.111984652065435D 01	0.111984652172219D 01	0.106783182296510D-08
0.500000000000000D 00	0.112837916728166D 01	0.112837916709551D 01	-0.186146431602197D-09
0.666666666666667D 00	0.110773216876300D 01	0.110773216743247D 01	-0.133053079665046D-08
0.750000000000000D 00	0.108806525357446D 01	0.108806525213102D 01	-0.144343803576419D-08
0.100000000000000D 01	0.999999999999999D 00	0.100000000000000D 01	0.105471187339390D-14

V. Numerical Evaluation of the Coefficients in the Taylor Series Expansion of $[\Gamma(z+1)]^{-1}$ About $z=0$

In this section we illustrate computation of the coefficients $h_{n,k}$ according to the prescription (49)–(51). The relation between the notation in the latter equations and the machine printouts is as follows.

This paper	Machine
$f_{n,k}$, $g_{n,k}$	F(N, K), G(N, K)
$h_{n,k}$	H(N, K)
z	Z
$\sum_{k=0}^{n} h_{n,k} z^k$	HN(Z)

Machine calculations were done for $n = 2(1)\,40$. The data for $n = 10$ and 15 are recorded in Table III, and we deduce that the coefficients $h_{n,k}$ are correct to at least 3 and 6 decimal places, respectively. For $n = 20$ and 30, the values of $h_{n,k}$ are correct to at least 8 and 13 decimal places, respectively. The quantity h_1 is γ, the Euler or Euler–Mascheroni constant. Let $\gamma_n = h_{n,1}$ be the approximation to γ. The values obtained for γ_n, $n = 5(5)\,35$ are given in Table IV. The value given there for $n = \infty$ is γ to 15 decimal places. Note that the difference of γ_n for $n = 30$ and 35 from the true γ is no doubt due to round off. Thus on a heuristic basis, we have an alternative scheme to obtain the Taylor series coefficients and in particular Euler's constant.

TABLE IV

Approximations for Euler's Constant

n	γ_n
5	0.57692 30769 23077
10	0.57722 04905 30812
15	0.57721 56570 84541
20	0.57721 56649 92483
25	0.57721 56649 00408
30	0.57721 56649 01537
35	0.57721 56649 01535
∞	0.57721 56649 01533

References

1. C. Lanczos, Trigonometric interpolation of empirical and analytic functions, *J. Math. Phys.* **17** (1938), 123–199.
2. C. Lanczos, "Applied Analysis," Prentice–Hall, Englewood Cliffs, N.J., 1956.
3. Y. L. Luke, "The Special Functions and Their Approximations," Vols. 1, 2, Academic Press, New York, 1969.
4. E. L. Ortiz, The tau method, *SIAM J. Numer. Anal.* **6** (1969), 480–492.
5. Y. L. Luke, "Mathematical Functions and Their Approximations," Academic Press, New York, 1975.